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Optimal Approximation of Periodic Analytic Functions with Integrable Boundary Values

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Let $S=\{z\in\mathbb{C}\colon |\mathrm{Im}(z)|<\beta\}$ be a strip in the complex plane. $\tilde{H}^q$, $1\leqslant q<\infty$, denotes the space of functions, which are analytic and 2π-periodic in S, real-valued on the real axis, and possess q-integrable boundary values. Let μ be a positive measure on $[0, 2\pi]$ and $\tilde{L}_p(\mu)$ be the corresponding Lebesgue space of periodic real-valued functions on the real axis. The even dimensional Kolmogorov, Gel'fand, and linear widths of the unit ball of $\tilde{H}^q$ in $\tilde{L}_p(\mu)$ are determined exactly, when $1\leqslant p\leqslant q<\infty$ or when $2=q<p<\infty$ and β is sufficiently large. It is shown that all three n-widths coincide and a characterization of the widths in terms of Blaschke products is established.

1. INTRODUCTION

Let $S=\{z\in\mathbb{C}\colon |\mathrm{Im}(z)|<\beta\}$ be a strip in the complex plane and let $\tilde{H}^q$, $1\leqslant q<\infty$, denote the space of functions f, which are analytic and 2π-periodic in S, real on the real axis, and satisfy $\sup_{-\beta<\eta<\beta}(1/4\pi)\int_0^{2\pi}|f(t+i\eta)|^q\,dt<\infty$. A function f in $\tilde{H}^q$ has a non-tangential limit almost everywhere on ∂S. The boundary function belongs to L_q and the norm

$$\|f\|_{\tilde{H}^q}:=\left(\frac{1}{4\pi}\int_0^{2\pi}|f(t+i\beta)|^q+|f(t-i\beta)|^q\,dt\right)^{1/q}$$

induces a Banach space structure on $\tilde{H}^q$. Further details on $\tilde{H}^q$ can be found in [Sar].

The present paper deals with optimal approximation of functions in $\tilde{H}^q$, where the term optimal approximation will be interpreted in the sense of n-widths. Hereby the following three classes of n-widths will be considered: Kolmogorov n-widths, Gel'fand n-widths and linear n-widths. We find the precise value of the even dimensional n-widths of the unit ball $\tilde{A}^q$ of $\tilde{H}^q$ in the target space $\tilde{L}_p$, when $1\leqslant p\leqslant q<\infty$ or when $2=q<p<\infty$ and β is

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sufficiently large. Here $\tilde{L}_p$ denotes the Lebesgue space $\tilde{L}_p([0, 2\pi], \mu)$ of periodic real valued functions defined on the real axis and μ is a positive measure on $[0, 2\pi]$. It turns out that all three kinds of n-widths coincide. Moreover sampling is optimal for $\tilde{A}^q$, i.e. there exists an optimal linear approximation operator based on point evaluations.

Similar investigations were already carried out by [FS1] and [FS2]. These authors determined the n-widths of the unit ball of the Hardy space $H^q(\Delta)$ in $L_p(E, \mu)$. Here Δ is the unit disk in the complex plane; E is a compact subset of Δ and μ a positive measure on E. The content of the present paper consists in studying how the approach of Fisher and Stessin can be extended and adapted from the nonperiodic to the periodic case. On the other side the present results extend results of [Wil1], [Wil2] and [Osi], where the even dimensional widths of $\tilde{A}^\infty$ in $\tilde{L}_p$ for $1 \leqslant p \leqslant \infty$ were determined. Finally one must definitely mention the fundamental pioneering paper of Fisher and Micchelli [FM], which elucidated the situation in the nonperiodic case for $q = \infty$ and $1 \leqslant p \leqslant \infty$ and served as stimulation for the other work mentioned above. It were Fisher and Micchelli, who for the first time pointed out the strong connection between Blaschke products and n-widths of Hardy spaces. Blaschke products will also be of central importance for the analysis of the present paper.

In Section 2 we fix our notation and formulate the main result, while Section 3 contains the corresponding proof.

2. The Main Result

The Kolmogorov n-widths of a subset A of a Banach space X are defined by

$$d_n(A, X) = \inf_{X_n} \sup_{x \in A} \inf_{y \in X_n} \|x - y\|,$$

where X_n runs over all subspaces of X of dimension n or less.

The Gel'fand n-widths of A in X are defined by

$$d^n(A, X) = \inf_{L^n} \sup_{x \in L^n \cap A} \|x\|,$$

where L^n runs over all subspaces of codimension at most n.

The linear n-widths of A in X are given by

$$\delta_n(A, X) = \inf_{P_n} \sup_{x \in A} \|x - P_n x\|,$$

where P_n is any linear operator of rank at most n mapping X into itself.

In the present paper we are interested in the even dimensional n-widths of $\tilde{A}^q$ in $\tilde{L}_p$. We say that sampling is optimal, if there exists an optimal linear approximation operator P_{2n} based on point evaluation in fixed points $z_1, ..., z_{2n}$ in $[0, 2\pi)$ combined with fixed functions $c_1, ..., c_{2n} \in \tilde{L}_p$:

$$(P_{2n} f)(z) = \sum_{k=1}^{2n} f(z_k)\, c_k(z), \qquad f \in \tilde{H}^q$$

$$\delta_{2n}(\tilde{A}^q, \tilde{L}_p) = \sup_{f \in \tilde{A}^q} \| f - P_{2n} f \|_{\tilde{L}_p}.$$

Our approach to the periodic function spaces $\tilde{H}^q$ will consist in transfering the analysis from the strip S to the annulus $\Omega = \{ w \in \mathbb{C}\colon\ R < |w| < 1/R \}$, where $R = e^{-\beta}$. The universal covering transformation $w = e^{iz}$ maps S onto Ω and the operator

$$I\colon f(z) \to g(w) = f\left(\frac{1}{i} \ln(w)\right)$$

yields an isometry between $\tilde{H}^q$ and $H^q(\Omega)$, the space of all functions g, which are analytic in Ω, real valued on the unit circle $E = \{ w \in \mathbb{C}\colon\ |w| = 1 \}$, and satisfy

$$\| g \|_{H^q} := \left(\frac{1}{4\pi} \int_0^{2\pi} |g(Re^{i\theta})|^q + \left| g\left(\frac{1}{R} e^{i\theta}\right) \right|^q d\theta \right)^{1/q} < \infty.$$

Since g is real valued on E, the reflection principle implies that $\overline{g(1/\bar{w})} = g(w)$. Therefore the H^q norm of g may be expressed as follows:

$$\| g \|_{H^q} = \left(\frac{1}{2\pi} \int_0^{2\pi} |g(Re^{i\theta})|^q \, d\theta \right)^{1/q}.$$

Furthermore I maps the space $\tilde{L}_p$ isometrically onto the corresponding space L_p, defined on the unit circle E. Denoting by A^q the unit ball in $H^q(\Omega)$, we see that the n-widths of $\tilde{A}^q$ in $\tilde{L}_p$ are equal to the n-widths of A^q in L_p. In the sequel we will concentrate exclusively on the later setting.

For the determination of the n-widths of A^q in L_p we need the notion of Blaschke products. A Blaschke product B of degree m on Ω is a function of the form

$$B(w) = \exp\left(- \sum_{j=1}^{m} (g(w, \alpha_j) + ih(w, \alpha_j)) \right).$$

Here $\alpha_1, ..., \alpha_m$ are points in Ω, $g(w, \alpha_j)$ is the Green's function for Ω with singularity at α_j and $h(w, \alpha_j)$ is the harmonic conjugate of $g(w, \alpha_j)$. In general B is multiple valued. However, if we choose $m = 2n$ and locate all points $\alpha_1, ..., \alpha_{2n}$ on the unit circle E, it turns out that B is single valued. For a proof of the last fact and further information on Blaschke products in multiply connected domains we refer to [Fis].

Finally we denote by $\mathscr{B}_{2n}$ the set of all Blaschke products of degree $2n$, all whose zeros lie on E. In view of the symmetry of Ω with respect to E, all functions in $\mathscr{B}_{2n}$ are real valued on E.

We are now prepared to formulate our main result.

THEOREM 1. *Suppose that $1 \leqslant p \leqslant q < \infty$ or that $q = 2$ and p satisfies*

$$\frac{1}{p} > \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \frac{1}{\cosh(2m+1)\,\beta}.$$

Then

$$d_{2n}(A^q, L_p) = d^{2n}(A^q, L_p) = \delta_{2n}(A^q, L_p) = \inf_{B \in \mathscr{B}_{2n}} \sup_{g \in A^q} \|gB\|_{L_p}.$$

Moreover, sampling is optimal for $\delta_{2n}(A^q, L_p)$.

3. PROOFS

As mentioned in the introduction our course of proof is similar to Fisher and Stessin. Although we emphasize here of course on the specific new features of the periodic case, some repetitions of arguments from Fisher and Stessin are inevitable in order to make the present paper self contained.

The starting point of our analysis is the following integral representation formula for real valued functions u, which are harmonic in Ω, continuous in $\bar{\Omega}$ and symmetric with respect to the unit circle, i.e. $u(z) = u(1/\bar{z})$. Let $p = e^{i\tau}$ be a fixed point on the unit circle. Then we have (see [Ach], p. 217):

$$u(p) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta})\, K_R(e^{i\theta}, p)\, d\theta, \tag{1}$$

where

$$K_R(e^{i\theta}, p) = K_R(e^{i\theta}, e^{i\tau}) = \frac{2\Lambda}{\pi} \operatorname{dn}\left(\frac{\Lambda}{\pi}(\theta - \tau), \lambda\right)$$

and

$$\frac{\pi \Lambda'}{\Lambda} = \beta.$$

Here $\mathrm{dn}(z, \lambda)$ denotes the Jacobi elliptic function with modulus λ (see for example [Bat]). The complementary modulus is given by $\lambda' = \sqrt{1-\lambda^2}$ and the complete elliptic integrals of the first kind with moduli λ and λ' are denoted by Λ and Λ', respectively.

The kernel $K_R(e^{i\theta}, p)$ is always positive. The representation formula (1) will be of great importance in the sequel, inasmuch as it is the adequate generalization of the Poisson integral formula to the doubly connected annulus.

A further main ingredient for the proof of Theorem 1 is the following extremal problem: for $1 \leqslant p$, $q < \infty$, and a measure μ on E define

$$\delta(p, q, \mu) := \sup\{ \|g\|_{L_p} / \|g\|_{H^q} : g \in H^q \}. \tag{2}$$

A compactness argument shows the existence of solutions of (2). Moreover we claim that any solution g is free of zeros. Suppose on the contrary that $g(z^*) = 0$ for $z^* \in \Omega$. Then by symmetry $g(1/\overline{z^*}) = 0$ as well. Division by the single valued Blaschke product of degree 2 with zeros in z^* and $1/\overline{z^*}$ leaves the H^q norm invariant, while strictly increasing the L_p norm. This contradiction shows that g must be zero free. We shall call a solution of (2) *normalized* if it has H^q norm one and is positive at the point $z = 1$. We claim that a normalized solution is uniquely determined, if the assumptions of Theorem 1 are satisfied. The proof of this assertion proceeds in two steps.

LEMMA 1. *Let g be a normalized solution of* (2). *Then*

$$\delta^p \, |g(Re^{i\theta})|^q = \int_E |g(w)|^p \, K_R(e^{i\theta}, w) \, d\mu(w)$$

for all $\theta \in [0, 2\pi]$, where δ is an abbreviation for $\delta(p, q, \mu)$.

Proof. Let u be a real harmonic function on Ω, which is continuous on $\bar{\Omega}$ and symmetric with respect to E. Let v denote the locally well defined harmonic conjugate function of u. The Cauchy–Riemann equations imply that for points w on E we have $(\partial v/\partial s)(w) = (\partial u/\partial n)(w) = 0$. The normal derivative $(\partial u/\partial n)$ vanishes on E because of the symmetry of u. Consequently the period of v along E is zero and the holomorphic function $u + iv$

is globally well defined on $\bar{\Omega}$. Set $f_\varepsilon = \exp(\varepsilon(u+iv))$, where ε is a small positive or negative parameter. Since $\delta \geqslant \| g f_\varepsilon \|_{L_p} / \| g f_\varepsilon \|_{H^q}$, we obtain that

$$\delta \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(Re^{i\theta})|^q \, e^{\varepsilon q u(Re^{i\theta})} \, d\theta \right\}^{1/q} \geqslant \left\{ \int_E |g(w)|^p \, e^{\varepsilon p u(w)} \, d\mu(w) \right\}^{1/p}.$$

Expanding the exponential terms and using the binomial theorem together with the fact that the solution g is normalized we conclude that

$$\delta^p \frac{1}{2\pi} \int_0^{2\pi} |g(Re^{i\theta})|^q \, u(Re^{i\theta}) \, d\theta = \int_E |g(w)|^p \, u(w) \, d\mu(w).$$

Expressing $u(w)$ by formula (1) yields

$$\begin{aligned} &\int_E |g(w)|^p \, u(w) \, d\mu(w) \\ &\quad = \int_E |g(w)|^p \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \, K_R(e^{i\theta}, w) \, d\theta \, d\mu(w) \\ &\quad = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \int_E |g(w)|^p \, K_R(e^{i\theta}, w) \, d\mu(w) \, d\theta. \end{aligned}$$

Application of the du Bois–Reymond lemma completes the proof of Lemma 1. ▮

LEMMA 2. *Suppose that* $1 \leqslant p \leqslant q < \infty$ *or that* $1 \leqslant q < p < \infty$ *and*

$$\frac{q}{p} > \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \frac{1}{\cosh(2m+1)\,\beta}. \tag{3}$$

Then a normalized solution of (2) *is uniquely determined.*

Proof. Let g_1 and g_2 be two normalized solutions of (2). Lemma 1 implies that

$$|g_1(Re^{i\theta})/g_2(Re^{i\theta})|^q = \int_E |g_1(w)/g_2(w)|^p \, |g_2(w)|^p \, K_R(e^{i\theta}, w) \, d\mu(w) \Bigg/ \int_E |g_2(w)|^p \, K_R(e^{i\theta}, w) \, d\mu(w).$$

Since $d\gamma(w) = |g_2(w)|^p \, K_R(e^{i\theta}, w) \, d\mu(w) / \int_E |g_2(w)|^p \, K_R(e^{i\theta}, w) \, d\mu(w)$ is a probability measure, we conclude that for each $\theta \in [0, 2\pi]$

$$|g_1(Re^{i\theta})/g_2(Re^{i\theta})|^q \leqslant \sup_{w \in E} |g_1(w)/g_2(w)|^p.$$

Setting $u = \ln |g_1/g_2|$ we may rephrase the last inequality in the form

$$\sup_{\theta \in [0, 2\pi]} u(Re^{i\theta}) \leqslant \frac{p}{q} \sup_{w \in E} u(w).$$

Both extremal functions g_1 and g_2 have no zeros in Ω. Consequently u is a harmonic function. Assuming that $1 \leqslant p \leqslant q < \infty$, we have $p/q \leqslant 1$ and the maximum principle implies that u is constant. Hence g_1 is a constant multiple of g_2 and the constant must be 1, since g_1 and g_2 are both normalized. This proves the first part of Lemma 2.

The case $q < p$ is somewhat more involved. Interchanging the role of g_1 and g_2 in the above analysis we obtain

$$-\inf_{\theta \in [0, 2\pi]} u(Re^{i\theta}) \leqslant -\frac{p}{q} \inf_{w \in E} u(w).$$

Combining this inequality with the corresponding inequality for the supremum yields

$$\sup_{\theta \in [0, 2\pi]} u(Re^{i\theta}) - \inf_{\theta \in [0, 2\pi]} u(Re^{i\theta}) \leqslant \frac{p}{q} (\sup_{w \in E} u(w) - \inf_{w \in E} u(w)).$$

Let us assume that u is not identically constant. Then after appropriately scaling u we may suppose that $-1 \leqslant u \leqslant 1$ and that the left-hand side of the last inequality equals 2. The right-hand side may be written in the form

$$\frac{p}{q} \sup_{\tau, \varphi \in [0, 2\pi]} (u(e^{i\tau}) - u(e^{i\varphi})).$$

Using the representation formula (1) we obtain:

$$2 \leqslant \frac{p}{q} \sup_{\tau, \varphi \in [0, 2\pi]} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |K_R(e^{i\theta}, e^{i\tau}) - K_R(e^{i\theta}, e^{i\varphi})| \, d\theta \right\}.$$

Let us denote the last supremum by M. We will show that

$$M = \frac{8}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \frac{1}{\cosh(2m+1)\,\beta}. \tag{4}$$

Since assumption (3) means that $M < 2q/p$, we arrive at a contradiction. Consequently u must be identically constant, which in turn implies the desired uniqueness in Lemma 2.

What remains to be done, is to establish identity (4). In view of the particular properties of the elliptic function dn we conclude that

$$
\begin{aligned}
M &= \sup_{\tau,\, \varphi \in [0,\, 2\pi]} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |K_R(e^{i\theta}, e^{i\tau}) - K_R(e^{i\theta}, e^{i\varphi})|\, d\theta \right\} \\
&= \frac{1}{2\pi} \int_0^{2\pi} |K_R(e^{i\theta}, 1) - K_R(e^{i\theta}, -1)|\, d\theta \\
&= \frac{\Lambda}{\pi^2} \int_0^{2\pi} \left| \operatorname{dn}\left(\frac{\Lambda}{\pi}\theta, \lambda\right) - \operatorname{dn}\left(\frac{\Lambda}{\pi}(\theta+\pi), \lambda\right) \right| d\theta \\
&= \frac{4\Lambda}{\pi^2} \int_0^{\pi/2} \operatorname{dn}\left(\frac{\Lambda}{\pi}\theta, \lambda\right) - \operatorname{dn}\left(\frac{\Lambda}{\pi}(\theta+\pi), \lambda\right) d\theta.
\end{aligned}
$$

Using the relation

$$
\operatorname{dn}\left(\frac{\Lambda}{\pi}\theta, \lambda\right) = \frac{\pi}{2\Lambda} + \frac{\pi}{\Lambda} \sum_{m=1}^{\infty} \frac{\cos m\theta}{\cosh m\beta}
$$

we obtain

$$
\frac{4\Lambda}{\pi^2} \int_0^{\pi/2} \operatorname{dn}\left(\frac{\Lambda}{\pi}\theta, \lambda\right) d\theta = 1 + \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \frac{1}{\cosh(2m+1)\,\beta}.
$$

Hence

$$
M = \frac{8}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \frac{1}{\cosh(2m+1)\,\beta}. \qquad \blacksquare
$$

Our next aim will be to establish the lower bound for $d_{2n}(A^q, L_p)$ and $d^{2n}(A^q, L_p)$. For this purpose we need in addition to Lemma 2 the following version of the Pick–Nevanlinna interpolation theorem for the space $H^\infty(\Omega, \mathbb{C})$, consisting of all complex valued bounded holomorphic functions on Ω. Functions in $H^\infty(\Omega, \mathbb{C})$ are not necessarily real valued on the unit circle E.

THEOREM 2. *Fix $2n+1$ distinct points $z_0, ..., z_{2n}$ in E and let $t_0, ..., t_{2n}$ be $2n+1$ real numbers with $\sum_{j=0}^{2n} |t_j|^2 = 1$, so that the vector $\mathbf{t} = (t_0, ..., t_{2n})$ belongs to the unit sphere S^{2n} of $\mathbb{R}^{2n+1}$. Set*

$$
\rho(\mathbf{t}) = \inf\{ \|f\|_{H^\infty} : f \in H^\infty(\Omega, \mathbb{C}),\ f(z_j) = t_j,\ 0 \leqslant j \leqslant 2n \}.
$$

Then we have:

(i) ρ *is a continuous function on* S^{2n}.

(ii) *There is a unique function* $B_{\mathbf{t}} \in H^{\infty}(\Omega, \mathbb{C})$ *with*

$$\|B_{\mathbf{t}}\| = 1 \qquad \text{and} \qquad B_{\mathbf{t}}(z_j) = t_j/\rho(\mathbf{t}), \qquad 0 \leqslant j \leqslant 2n.$$

(iii) *The function* $B_{\mathbf{t}}$ *is a single valued Blaschke product of degree at most* $2n+1$.

(iv) $\sigma\colon \mathbf{t} \to B_{\mathbf{t}}$ *is a continuous mapping from* S^{2n} *into the set* $\mathscr{B}_{2n+1}$ *of single valued Blaschke products of degree at most* $2n+1$, *when* $\mathscr{B}_{2n+1}$ *is endowed with the topology of locally uniform convergence on* Ω.

For a detailed exposition of the Pick–Nevanlinna theorem we refer to [Fis], Chapter 5.

A priori it is possible that the Blaschke product $B_{\mathbf{t}} \in \mathscr{B}_{2n+1}$ interpolating the data $t_j/\rho(\mathbf{t})$ with minimal H^{∞} norm is complex valued on E. However, since the data $\mathbf{t}$ are real, the Schwarz reflection principle implies that $\overline{B_{\mathbf{t}}(1/\bar{z})}$ is a minimal interpolant as well. In view of uniqueness we conclude that $B_{\mathbf{t}}$ is real valued on E and its zeros are located symmetric with respect to E. Let $z_1, ..., z_k$ be the zeros of $B_{\mathbf{t}}$, counting multiplicities. Then the period of $\arg(B_{\mathbf{t}})$ along E is given by $2\pi \sum_{j=1}^{k} \omega(z_j)$, where ω is the unique harmonic function with constant boundary values 1 and 0 on the inner and outer boundary of Ω, respectively. Hence $B_{\mathbf{t}}$ is single valued if and only if $\sum_{j=1}^{k} \omega(z_j) \in \mathbb{N}$. Since $\omega(\zeta) + \omega(1/\bar{\zeta}) = 1$ for all $\zeta \in \Omega$ and in particular $\omega(\zeta) = \frac{1}{2}$ for $\zeta \in E$, the condition $\sum_{j=1}^{k} \omega(z_j) \in \mathbb{N}$ implies that the degree k of $B_{\mathbf{t}}$ must be even.

Let us denote by $\hat{\mathscr{B}}_{2n}$ the set of all Blaschke products with an even number of zeros less or equal $2n$, which are located on E or symmetric with respect to E. As a result of the preceding analysis we obtain an odd continuous mapping

$$\sigma\colon S^{2n} \to \hat{\mathscr{B}}_{2n}, \qquad \mathbf{t} \mapsto B_{\mathbf{t}}.$$

We now use the map σ, in order to construct an odd continuous mapping τ from S^{2n} into A^q: For each Blaschke product $B \in \hat{\mathscr{B}}_{2n}$, let g_B be the unique normalized solution of (2) with respect to the measure $|B|^p \, d\mu$ and define

$$\tau(\mathbf{x}) = \sigma(\mathbf{x}) \, g_{\sigma(\mathbf{x})}, \qquad \mathbf{x} \in S^{2n}.$$

Having established the existence of τ, we use now the same technique based on Borsuk's theorem like [FS1] to conclude that

$$\inf_{B \in \hat{\mathscr{B}}_{2n}} \sup_{g \in A^q} \|gB\|_{L_p} \leqslant d_{2n}(A^q, L_p), d^{2n}(A^q, L_p).$$

What remains to be done is to show that

$$\inf_{B \in \hat{\mathscr{B}}_{2n}} \sup_{g \in A^q} \|gB\|_{L_p} = \inf_{B \in \mathscr{B}_{2n}} \sup_{g \in A^q} \|gB\|_{L_p}.$$

The last equality is implied by the following two observations:

(i) Let $B \in \hat{\mathscr{B}}_{2n}$ be a Blaschke product with less than $2n$ zeros: Then multiplying B with a symmetric Blaschke product of degree 2 yields a function in $\hat{\mathscr{B}}_{2n}$ and reduces the norm $\|gB\|_{L_p}$.

(ii) Let $B \in \hat{\mathscr{B}}_{2n}$ possess a symmetric pair of zeros z_1 and $z_2 = 1/\bar{z}_1$, which does not ly on E. Let z_0 be the orthogonal projection of z_1 onto E. Replacing z_1 and z_2 by a double zero in z_0 yields a Blaschke product $B_0 \in \mathscr{B}_{2n}$ such that $\|gB_0\|_{L_p} \leqslant \|gB\|_{L_p}$.

Altogether this completes the proof of the lower bound for $d_{2n}(A^q, L_p)$ and $d^{2n}(A^q, L_p)$.

We now turn to the upper bound. Since the Kolmogorov and Gel'fand widths are always less or equal than the linear widths (see [Pin]), we may confine ourselve to proving that

$$\delta_{2n}(A^q, L_p) \leqslant \inf_{B \in \mathscr{B}_{2n}} \sup_{f \in A^q} \|fB\|_{L_p}.$$

First we consider the case $p \leqslant q$. Let B be a Blaschke product in $\mathscr{B}_{2n}$ with zeros $z_1, ..., z_{2n}$ and let g_B be the unique normalized solution of (2) for the measure $|B|^p \, d\mu$. Define T: $H^q \to L_p$ by

$$Tf(z) = B(z)\, g_B(z) \frac{1}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\theta})}{B(Re^{i\theta})\, g_B(Re^{i\theta})} K_R(z, e^{i\theta})\, d\theta$$

where the kernel $K_R(z, e^{i\theta})$ is given by (1). Then $Th = h$ for functions h vanishing at the zeros of B. In general, let $\Phi_1, ..., \Phi_{2n}$ be $2n$ bounded periodic analytic functions on Ω, which satisfy $\Phi_i(z_j) = \delta_{ij}$. Observing that the difference $f - \sum_{j=1}^{2n} f(z_j)\, \Phi_j$ has zeros in $z_1, ..., z_{2n}$, we obtain that

$$Tf = f - \sum_{j=1}^{2n} f(z_j)(\Phi_j - T\Phi_j).$$

Hence T has the form $T = I - P$, where P is a linear sampling operator of rank less or equal $2n$. Consequently $\delta_{2n}(A^q, L_p)$ is bounded from above by $\|T\|$. In order to estimate $\|T\|$, we may proceed similarly like [FS2]. However, since we are dealing with the annulus instead of the unit disk, we must replace the Poisson kernel by the kernel $K_R(e^{i\theta}, z)$ defined in (1). In this way we obtain:

$$\delta_{2n}(A^q, L_p) \leqslant \sup_{g \in A^q} \|gB\|_{L_p}.$$

Because B is an arbitrary Blaschke product in $\mathscr{B}_{2n}$, this yields the desired upper estimate of $\delta_{2n}(A^q, L_p)$ for $1 \leqslant p \leqslant q < \infty$.

What remains to be done, is to establish the case $2 = q < p < \infty$. Since H^2 is a Hilbert space, it is known that $\delta_{2n}(A^2, L_p) = d^{2n}(A^2, L_p)$ (see [Pin]). Considering the $2n$-codimensional subspace $\{f \in H^2 : f(z_1) = \cdots = f(z_{2n}) = 0\}$ we see that $d^{2n}(A^2, L_p) \leqslant \sup_{g \in A^2} \|gB\|_{L_p}$. Taking the infimum over all B in $\mathscr{B}_{2n}$ completes the proof of the upper bound of Theorem 1.

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References

[Ach] N. I. Achieser, "Theory of Approximation," Ungar, New York, 1956.
[Bat] Bateman Manuscript Project, "Higher Transcendental Functions," Vol. II, McGraw–Hill, New York, 1953.
[Fis] S. D. Fisher, "Function Theory on Planar Domains," Wiley, New York, 1983.
[FM] S. D. Fisher and C. A. Micchelli, The n-widths of sets of analytic functions, *Duke Math. J.* **47** (1980), 789–801.
[FS1] S. D. Fisher and M. I. Stessin, The n-width of the unit ball of H^q, *J. Approx. Theory* **67** (1991), 347–356.
[FS2] S. D. Fisher and M. I. Stessin, The n-width of the unit ball of H^q-addendum, *J. Approx. Theory* (1994), to appear.
[Osi] K. Yu. Osipenko, On n-widths, optimal quadrature formulas and optimal recovery of functions analytic in a strip, *Izv. Russian Akad. Nauk Ser. Mat.* (1994), to appear.
[Pin] A. Pinkus, "n-Widths in Approximation Theory," Springer-Verlag, Berlin, 1985.
[Sar] D. Sarason, The H^p spaces of an annulus, *Memoirs Amer. Math. Soc.* **56** (1965).
[Wil1] K. Wilderotter, "Optimale Algorithmen zur Approximation analytischer Funktionen," dissertation, Bonn, 1990.
[Wil2] K. Wilderotter, On n-widths of bounded periodic holomorphic functions, *Ukrainian Math. J.* (1994), to appear.

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